

# Witness for an entangled state with positive partial transpose

Nirman Ganguly <sup>1,2\*</sup>, Binayak.S.Choudhury <sup>2</sup>

<sup>1</sup> Heritage Institute of Technology, Kolkata-107, West Bengal, India

<sup>2</sup> Bengal Engineering and Science University,  
Shibpur,Howrah, West Bengal, India

## Abstract

In this letter we have analyzed an entangled state in  $C^3 \otimes C^3$  having a positive partial transposition and have shown that it is an edge state. Further we have constructed explicitly a witness operator  $W$  which detects the entanglement.

PACS numbers: 03.67.-a

## 1 Introduction

Quantum entanglement , first recognized by Einstein, Podolsky and Rosen [1]and Schrodinger [2], is one of the most amazing features of quantum formalism. After over seventy years it is still a fascinating object from both theoretical and experimental points of view . Together with the development of its knowledge theoretically and experimentally a number of practical applications including quantum computation [3] and quantum teleportation [4]have been shown .

---

\*Corresponding author: nirmanganguly@rediffmail.com

Given the vast importance of entanglement , what becomes very relevant are methods to distinguish between entangled and separable states. For low dimensional ( $2 \otimes 2$  and  $2 \otimes 3$ ) states there exist simple necessary and sufficient conditions for separability [5,6] which is based on the fact that separable states have a positive partial transpose. For higher dimensional systems all states with negative partial transpose(NPT) are entangled . But a very intriguing fact is the existence of entangled states in higher dimension having positive partial transpose [7,8]. Thus the separability problem boils down to finding whether density operators with positive partial transpose (PPT) are entangled or not.

Of specific importance in this context are the so called edge states [9]. A PPT entangled state  $\varrho$  is called an edge state if for any  $\varepsilon > 0$  and any product vector  $|e, f\rangle$ ,  $\varrho' = \varrho - \varepsilon|e, f\rangle$  is not a PPT state. Thus edge states lies at the boundary of PPT and NPT states. An interesting character that an edge state shows is extreme violation of the range criterion namely there exists no product vector  $|e, f\rangle$  belonging to the range of the edge state  $\varrho$  such that  $|e, f^*\rangle$  (conjugation is done with respect to the second system) belongs to the range of  $\varrho^{T_B}$  [9].

A very general method to distinguish between entangled and separable states is through witness operators and positive maps [6,10]. An entanglement witness  $W$  is an operator for which  $\text{Tr}(W\rho) < 0$  for entangled states  $\rho$  and  $\text{Tr}(W\sigma) \geq 0$  for separable states  $\sigma$  [6,10]. Entanglement witness operators can be categorized as decomposable and non-decomposable. A decomposable entanglement witness  $W$  is an operator which can be written as  $W = P + Q^{T_B}$ , where  $P$  and  $Q$  are positive operators and transposition is done with respect to the second system. Witnesses which cannot be expressed in the above mentioned form are called non-decomposable[11].

A map  $\Lambda : B(H_A) \longrightarrow B(H_B)$  is called positive if it maps positive operators to positive operators, where  $B(H_A)$  and  $B(H_B)$  are sets of operators in Hilbert spaces  $H_A$  and  $H_B$  respectively. A positive map is called completely positive if any tensor extension of the map is also positive. A positive map is non-decomposable if it cannot be expressed in the form  $\Lambda = \Lambda_1^{CP} + \Lambda_2^{CP} T$ , where  $\Lambda_i^{CP}$  is a complete positive map. In [6] the separability problem was solved for  $2 \otimes 2$  and  $2 \otimes 3$  systems using decomposable maps. Thereafter in [10] a family of non-decomposable maps were introduced for higher dimensions as de-

composable maps fail to recognize entangled states in those dimensions.

In [9] it was shown that every PPTES(positive partial transpose entangled state)  $\rho$  can be expressed as  $\rho = (1 - \lambda)\rho_{sep} + \lambda\delta$  , where  $0 < \lambda < 1$ ,  $\rho_{sep}$  is a separable state and  $\delta$  is an edge state. Further it was shown in [12] that only a non-decomposable witness can detect a PPTES. In [7] a class of PPTES in  $C^2 \otimes C^4$  was introduced. It was shown to be an edge state and an entanglement witness was constructed for the same in [12].

In this letter we take the other example in  $C^3 \otimes C^3$  from [7] and construct an entanglement witness for it. We first prove that it is an edge state and thereby construct the witness to detect it. We also show that the said edge state state violate the range criterion [7] in an extreme manner. Thus our work paves way for extensive work on detection of entanglement. Our work is organized as follows :

In section 2 we prove the qutrit-qutrit state in [7] to be an edge state using procedures given in [9,12]. In section 3 we show the extreme violation of the range criterion by the edge state. In section 4 we construct the entanglement witness for the state and give it in the matrix form [9,12]. We end the letter with conclusions.

## 2 Evidence in support of edge state

In this section we prove that the state in  $C^3 \otimes C^3$  given in [7] is an edge state. We subtract a product vector from the state and show that the resultant state is not a PPT. We use a step by step approach.

STEP 1: We start with the density matrix and its partial transpose:

$$\rho_a = \frac{1}{8a+1} \begin{pmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{pmatrix} \quad (1)$$

$$\rho_a^{T_B} = \frac{1}{8a+1} \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & a & 0 & 0 \\ 0 & a & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & a & 0 \\ 0 & 0 & a & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & a & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{pmatrix} \quad (2)$$

where  $0 < a < 1$ .

STEP 2: We identify one of the product vectors present in the range of  $\rho_a$  as  $\frac{a}{8a+1}|12\rangle\langle 12|$ . Subtracting this vector from  $\rho_a$  and taking the partial transpose we get :

$$(\rho_a - \frac{a}{8a+1} |12\rangle\langle 12|)^{T_B} = \frac{1}{8a+1} \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & a & 0 & 0 \\ 0 & a & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & a & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & a & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{pmatrix} \quad (3)$$

STEP 3: We find the eigenvalues of the matrix given in (3). One of the eigenvalues of the matrix is  $\frac{\frac{1}{2}-\frac{1}{2}\sqrt{5}}{8a+1}a$ , which being negative proves that  $\rho_a$  is an edge state [9].

### 3 Violation of the range criterion

Here we show that the edge state given in equation (1) violates the range criterion extremely. We denote the range of  $\rho_a$  and  $\rho_a^{T_B}$  by  $R(\rho_a)$  and  $R(\rho_a^{T_B})$  respectively. A methodical approach is followed below:

STEP 1: The kernel of a density matrix  $\rho$  is defined as

$$\ker(\rho) = \{|x\rangle \in H_A : \rho|x\rangle = 0\}.$$

We enumerate the basis vectors belonging to the kernel of  $\rho_a$  and  $\rho_a^{T_B}$  below:

(i) Vectors in the kernel of  $\rho_a$ :  $|00\rangle + \frac{\sqrt{1-a^2}}{1+a}|20\rangle - |22\rangle$ ,  $|11\rangle + \frac{\sqrt{1-a^2}}{1+a}|20\rangle - |22\rangle$

(ii) Vectors in the kernel of  $\rho_a^{T_B}$ :  $-\frac{\sqrt{1-a^2}}{a-1}|02\rangle + \frac{\sqrt{1-a^2}}{a-1}|20\rangle + |22\rangle$ ,  $-|12\rangle + |21\rangle$ ,  $-|01\rangle + |10\rangle$

STEP 2: Without loss of generality we take an arbitrary product vector  $|e, f\rangle \in R(\rho_a)$  as,

$$|e, f\rangle = (x_1|0\rangle + y_1|1\rangle + z_1|2\rangle) \bigotimes (x_2|0\rangle + y_2|1\rangle + z_2|2\rangle) \quad (4)$$

such that  $|e, f^*\rangle \in R(\rho_a^{T_B})$

$$|e, f^*\rangle = (x_1|0\rangle + y_1|1\rangle + z_1|2\rangle) \bigotimes (x_2^*|0\rangle + y_2^*|1\rangle + z_2^*|2\rangle) \quad (5)$$

where conjugation is done with respect to the second system and  $x_i, y_i$  and  $z_i$  are complex numbers.

STEP 3: We take the inner product of the vectors in equation (4) with the vectors in the kernel of  $\rho_a$ . Similarly we take the inner product of the vectors in equation (5) with the vectors in the kernel of  $\rho_a^{T_B}$ . The inner products being zero yields a system of equations. This system gives a trivial solution for  $x_i, y_i$  and  $z_i$

i.e either  $x_1 = y_1 = z_1 = 0$  or  $x_2 = y_2 = z_2 = 0$  or both. Thus, there is no such  $|e, f\rangle \in R(\rho_a)$ , such that  $|e, f^*\rangle \in R(\rho_a^{T_B})$ . This again buttresses the fact that  $\rho_a$  is an edge state [9].

## 4 Construction of the Witness operator

In this section we construct an entanglement witness for the edge state as given in (1) . As given in [9] a non-decomposable witness to detect a PPTES is of the form

$$W = P + Q^{T_B} - \varepsilon I \quad (6)$$

where  $P$  can be taken to be a projector on the kernel of  $\rho_a$  and  $Q$  a projector on the kernel of  $\rho_a^{T_B}$  and  $\varepsilon = \inf_{|e, f\rangle} \langle e, f | P + Q^{T_B} | e, f \rangle$ .

The projector on the kernel of  $(\rho_a)$  is:

$$\begin{aligned} P = & |00\rangle\langle 00| + c|00\rangle\langle 20| - |00\rangle\langle 22| + c|20\rangle\langle 00| + \\ & c^2|20\rangle\langle 20| - c|20\rangle\langle 22| - |22\rangle\langle 00| - c|22\rangle\langle 20| + \\ & |22\rangle\langle 22| + |11\rangle\langle 11| + c|11\rangle\langle 20| - |11\rangle\langle 22| + \\ & + c|20\rangle\langle 11| + c^2|20\rangle\langle 20| - c|20\rangle\langle 22| - |22\rangle\langle 11| + \\ & - c|22\rangle\langle 20| + |22\rangle\langle 22| \end{aligned} \quad (7)$$

The partial transpose of the projector on the kernel of  $\rho_a^{T_B}$  is:

$$\begin{aligned} Q^{T_B} = & d^2|02\rangle\langle 02| - d^2|00\rangle\langle 22| - d^2|02\rangle\langle 22| - d^2|22\rangle\langle 00| \\ & + d^2|20\rangle\langle 20| + d|22\rangle\langle 20| - d|22\rangle\langle 02| + d|20\rangle\langle 22| \\ & + |22\rangle\langle 22| + |12\rangle\langle 12| - |11\rangle\langle 22| - |22\rangle\langle 11| \\ & + |21\rangle\langle 21| + |01\rangle\langle 01| - |00\rangle\langle 11| - |11\rangle\langle 00| + |10\rangle\langle 10| \end{aligned} \quad (8)$$

where  $c = \frac{\sqrt{1-a^2}}{1+a}$  and  $d = \frac{\sqrt{1-a^2}}{a-1}$ . We obtain  $\varepsilon = \inf_{|e,f\rangle} \langle e, f | P + Q^{T_B} | e, f \rangle = \frac{a^2}{(8a+1)^2}$ .

Hence, the entanglement witness is obtained as :

$$W = \begin{pmatrix} 1-\varepsilon & 0 & 0 & 0 & -1 & 0 & c & 0 & -(d^2 + 1) \\ 0 & 1-\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d^2 - \varepsilon & 0 & 0 & 0 & 0 & 0 & -d^2 \\ 0 & 0 & 0 & 1-\varepsilon & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1-\varepsilon & 0 & c & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1-\varepsilon & 0 & 0 & 0 \\ c & 0 & 0 & 0 & c & 0 & 2c^2 + d^2 - \varepsilon & 0 & d - 2c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\varepsilon & 0 \\ -(d^2 + 1) & 0 & -d & 0 & -2 & 0 & d - 2c & 0 & 3 - \varepsilon \end{pmatrix} \quad (9)$$

With the matrix  $W$  thus obtained we get the result

$\text{Tr}(W\rho_a) = -\varepsilon = -\frac{a^2}{(8a+1)^2}$ , thus detecting  $\rho_a$  as an entangled state.

For completeness we must mention the following two observations:

- (1) The witness (9) was constructed using methods given in [9]. However the witness is not optimal [12] as can be easily seen by subtracting the projector  $\frac{1}{2}|00\rangle\langle 00|$  from it and still obtaining another witness.
- (2) The approach used here to construct the witness as given in [9] is analytic in nature and different from the approach in [13]. The method used in [13] is numerical and use semi-definite programs to obtain an witness which arises as a feasible solution to the dual semi-definite program.

We thus, in this letter have shown that the class of states in  $C^3 \otimes C^3$  as proposed in [7] is an edge state. Further we have shown extreme violation of the range criterion by the state. We have obtained an entanglement witness to detect the entanglement of the said state.

## 5 Acknowledgement

NG acknowledges his mother for her eternal love and blessings.

## References

- [1] A.Einstein,B.Podolsky and N.Rosen, Phys Rev 47(1935),777.
- [2] E.Scrodinger, Naturewissenschaften 23(1935) 807.
- [3] A.Barenco, D.Deutsch,A.Ekert and R.Jozsa, Phys Rev Lett 74(1995) 4083.
- [4] C.Benett,G.Brassard,C.Crepeau,R.Jozsa,A.peres and W.K.Wootters, Phys Rev Lett 70(1993) 1895.
- [5] A.Peres, Phys Rev Lett 77,1413(1996).
- [6] M.Horodecki,P.Horodecki and R.Horodecki, Phys Lett A 223,1-8(1996).
- [7] P.Horodecki ,Phys Lett A 232,333(1997).
- [8] C.H.Bennett,D.P.Vincenzo,T.Mor,P.W.Shor , J.A.Smolin and B.M.Terhal ,Phys Rev Lett 82 5385(1999).
- [9] M.Lewenstein,B.Krauss,P.Horodecki and J.I.Cirac,arxiv:quant-ph/0005112v1(2000).
- [10] B.M.Terhal, arxiv: quant-ph/9810091.
- [11] S.L.Woronowicz,Rep.Math.Phys.10,165(1976)
- [12] M.Lewenstein,B.Krauss,J.I.Cirac and P.Horodecki,arxiv: quant-ph/0005014.
- [13] A.C.Doherty , P.A. Parrilo and F.M.Spedalieri, PRA 69, 022308 (2004).